First-Order Differential Equations

There exist several methods for solving first-order differential equations, and each pertains to a subclass of first-order equations. The method of integrating factor is used for solving first-order **linear** equations in the standard form,

$$P(t)y' + Q(t)y = G(t), \quad (1$$

where $P, Q, \& \, G$ are given constants or functions of the independent variable. Re-visit previous chapters for more information on linearity and order of differential equations.

Linear Differential Equations: Method of Integrating Factor

Some first-order differential equations can be solved immediately by integrating the equation, such as the following:

$$(4+t^2)y' + 2ty = 4t \quad (2$$

The left-hand side of equation (2) is a linear combination that also appears in the product rule for derivatives:

$$(4+t^2)y' + 2ty = \frac{d}{dt}((4+t^2)y) \quad (3$$

From equation (3), equation (2) can now be rewritten as

$$\frac{d}{dt}\big((4+t^2)y\big) = 4t \quad (4$$

Thus, by integrating both sides with respect to t, we obtain,

$$(4+t^2)y = 2t^2 + C \quad (5 \qquad \rightarrow \qquad y = {2t^2}/{4+t^2} + {C}/{4+t^2} \quad (6$$

This is the general solution of equation (2). However, this is almost never the case. Thus, we must find an integrating factor $u(t)$ that, when distributed, makes the equation immediately integrable using the product rule. From this, we obtain two solution methods.

Example:

Solve the differential equation $y' + {1}/{2}\,y = {1}/{2}\,e^{t/3}$ by method of integrating factor.

Method 1: Writing the Whole Story

I) STD Form $y' + {}^1/_2\, y = {}^1/_2\, e^{t/3}$

II) Distribute $u \rightarrow uy' + {}^1/_2\, uy = {}^1/_2\, ue^{t/3}$, then make LHS

$\rightarrow (uy)' = {}^1/_2\, ue^{t/3}$ **(1) [mark for later]**

III) Then, make RHS

$\rightarrow uy' + {}^1/_2\, uy = u'y + uy'$ **[cancel $u'y$ and y on both sides]**

$\rightarrow {}^1/_2\, u = u'$ **[solve the separable problem and find u]**

$\rightarrow u = Ce^{{}^1/_2 t}$ **[drop C's after integrating]**

IV) Plug u in **(1)** and solve for y

$\rightarrow \int (e^{{}^1/_2 t} y)' = {}^1/_2 \int e^{5t/6}\, dt \rightarrow e^{{}^1/_2 t} y = {}^3/_5\, e^{{}^5/_6 t} + C$ **[solve for y]**

V) The general solution is $y = {}^3/_5\, e^{{}^1/_3 t} + Ce^{-{}^1/_2 t}$

Method 2: Formula for *u(t)*

I) STD Form $y' + {}^1/_2\, y = {}^1/_2\, e^{t/3}$

II) Find $u = e^{\int P(t)dt}$

$\rightarrow = e^{\int {}^1/_2 dt} \rightarrow e^{{}^1/_2 t}$

III) Distribute $u = e^{{}^1/_2 t}$

$\rightarrow e^{{}^1/_2 t} y' + {}^1/_2\, e^{{}^1/_2 t} y = {}^1/_2\, e^{{}^5/_6 t}$

IV) Make LHS $(e^{{}^1/_2 t} y)' = {}^1/_2\, e^{{}^5/_6 t}$ and solve

$\rightarrow \int (e^{{}^1/_2 t} y)' = {}^1/_2 \int e^{{}^5/_6 t}\, dt \rightarrow e^{{}^1/_2 t} y = {}^3/_5\, e^{{}^5/_6 t} + C$

V) The general solution is $y = {}^3/_5\, e^{{}^1/_3 t} + Ce^{-{}^1/_2 t}$.

References:

Boyce, W.E. and DiPrima, R.C. (2022). *Elemntary Differential Equations*, (12). John Wiley & Sons. ISBN-13: 978-1119781981.

Disclaimer: We did not include all of the resources conferred to formulate this handout. We encourage students to conduct further research to find additional resources. The format of this list is not commensurate with a standard format.