Finding Inverses of a 3x3 Matrix

The inverse of a matrix can be used to solve a system of linear equations. Although a formula exists to find the inverse of a 2x2 matrix, 3x3 matrices and higher must use a process. The process to find the inverse of a 3x3 matrix involves the **coefficient matrix**, **the identity matrix,** and **the use of Gauss-Jordan elimination**.

Steps for Finding Inverses of a 3x3 Matrix

Step 1: Before finding the multiplicative inverse of the coefficient matrix, we must find the **coefficient matrix** itself. To do that, we convert a given linear system into an augmented matrix as shown in the example below.

$$\begin{cases} 2x + 3y + z = 4 \\ 3x + 3y + z = 5 \\ 2x + 4y + z = 6 \end{cases} \longrightarrow \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

System of Linear Equations — Coefficient Matrix — Constant Matrix

Step 2: Once we have our coefficient matrix, we can begin to find the **multiplicative inverse**. To find the inverse, we will combine the coefficient matrix and the 3x3 identity matrix, as shown below.

$$\left[\begin{matrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{matrix}\right. \left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]$$

Coefficient Matrix — Identity Matrix

Step 3: Once we setup the combined matrix, we will use **Gauss-Jordan Elimination** to make the left-hand side into an identity matrix. Our only focus will be the *left-hand side*, and whatever occurs to the right-hand side will be inconsequential. After the left-hand side becomes an identity matrix, whatever occurred on our right-hand side is now our inverse matrix.

Gauss-Jordan Steps	Combined Matrix result
Original	$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$-R_1 + R_2 = R_2$	$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$R_1 \leftrightarrow R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$-R_2 + R_3 = R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$R_3 \leftrightarrow R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
$-2R_1 + R_3 = R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -2 & 0 \end{bmatrix}$
$-3R_2 + R_3 = R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$

As seen in the final step of the table, our left-hand side of the matrix is now an identity matrix. This means that whatever remains on our right-hand side is the inverse matrix. Therefore, the inverse matrix of our original linear system is $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$.

References:

Abramson, J. P. (2021). *College Algebra*. OpenStax, Rice University.

Disclaimer: We did not include all the resources conferred to formulate this handout. We encourage students to conduct further research to find additional resources. The format of this list is not commensurate with a standard format.